

Hyperconfluent third-order supersymmetric quantum mechanics

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Abstract

The hyperconfluent third-order supersymmetric quantum mechanics, in which all the factorization energies tend to a common value, is analyzed. It will be shown that the final potential as well can be achieved by applying consecutively a confluent second-order and a first-order SUSY transformations, both with the same factorization energy. The technique will be applied to the free particle and the Coulomb potential.

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1 Introduction

Nowadays, *Supersymmetric Quantum Mechanics* (SUSY QM) has become the standard technique for generating solvable potentials from a given initial one [1–6]. Moreover, it represents a powerful tool for designing Hamiltonians with a fixed prescribed spectrum (see, e.g. [7–9]). In its higher-order version, in which the differential intertwining operators are of order greater than one, it is well known that several seed solutions of the initial stationary Schrödinger equation with an appropriate behavior are in general required for calculating the new potential, the eigenfunctions of the corresponding Hamiltonian, etc. [10–18]. If for some reason just one of those seeds is available, one is driven to what could be called as *hyperconfluent higher-order SUSY QM*, in which all the factorization energies which are involved tend to a common value.

In the past several works dealing with the *confluent second-order SUSY QM* have been elaborated [19–22]. Up to our knowledge, however, the *hyperconfluent third-order SUSY QM* has not been addressed explicitly. Of course, several papers involving the third-order SUSY QM have been done, but they are centered mainly in the case when the factorization energies are all different (see e.g. [23–29] and references therein).

In this article we aim to fill the gap by studying in detail the hyperconfluent third-order SUSY QM. In order to achieve this, we have arranged the paper as follows. In the next Section we will briefly review the confluent second-order SUSY QM. In Section 3 we will analyze the direct

approach to the hyperconfluent third-order SUSY QM, while in Section 4 we will address the corresponding iterative method. Section 5 explores the requirements that the seed solution has to obey in order to produce non-singular transformations as well as the eigenfunctions of the SUSY generated Hamiltonians. In Section 6 we will illustrate our general treatment by means of two specific examples, the free particle and the Coulomb potential. Our conclusions will be presented in Section 7.

2 Confluent second-order SUSY QM

Let us consider a one-dimensional Schrödinger Hamiltonian

$$H_0 = -\frac{d^2}{dx^2} + V_0(x). \quad (1)$$

The domain of definition or the corresponding system is denoted as $\mathcal{D} = [x_l, x_r]$. Thus, depending on the problem we are dealing with, and the consequent identification of x_l and x_r , this domain could be the full real line, the positive semi-axis or a finite interval. The eigenfunctions and eigenvalues associated to the discrete part of the spectrum of H_0 , denoted by $\psi_n(x)$, E_n , $n = 0, 1, \dots$, satisfy the stationary Schrödinger equation

$$H_0\psi_n = -\psi_n'' + V_0\psi_n = E_n\psi_n, \quad (2)$$

as well as the boundary conditions

$$\psi_n(x_l) = \psi_n(x_r) = 0. \quad (3)$$

From now on we are going to suppose that all the eigenfunctions and eigenvalues of H_0 are known. In the general formulation of the second-order SUSY QM one looks for a new Hamiltonian

$$H_2 = -\frac{d^2}{dx^2} + V_2(x), \quad (4)$$

which is intertwined with H_0 by a second-order operator B_2^+ in the way

$$H_2B_2^+ = B_2^+H_0, \quad (5)$$

where

$$B_2^+ = \frac{d^2}{dx^2} - \eta(x)\frac{d}{dx} + \gamma(x). \quad (6)$$

By plugging these expressions in the intertwining relationship (5), decoupling the resulting system of equations and solving it we arrive at [5, 6]:

$$V_2 = V_0 - 2\eta', \quad (7)$$

$$\gamma = \frac{\eta'}{2} + \frac{\eta^2}{2} - V_0 + \frac{\epsilon_1 + \epsilon_2}{2}, \quad (8)$$

$$\eta = \{\ln[W(u_1, u_2)]\}', \quad (9)$$

where u_1, u_2 are two seed solutions of the initial stationary Schrödinger equation associated to the factorization energies ϵ_1, ϵ_2 (in general different)

$$H_0u_i = -u_i'' + V_0u_i = \epsilon_iu_i, \quad i = 1, 2, \quad (10)$$

and $W(u_1, u_2) = u_1 u_2' - u_1' u_2$ denotes their Wronskian. Note that the seeds u_1, u_2 could obey or not the boundary conditions of equation (3).

The *confluent second-order SUSY QM* arises now as a limit procedure of the previous formalism when $\epsilon_1 \rightarrow \epsilon_2 \rightarrow \epsilon$ [19]. Note that, if the potential V_2 is going to be different from the initial one, then u_1 and u_2 cannot be just chosen as two linearly independent solutions of equation (10), since then $W(u_1, u_2) = \text{constant}$ and therefore $V_2 = V_0$. In order to produce non-trivial results, the right choice is to take u_1 as a standard eigenfunction of H_0 while u_2 becomes a generalized eigenfunction of rank 2 of H_0 , both associated to ϵ , namely:

$$(H_0 - \epsilon)u_1 = 0 \quad \Rightarrow \quad u_1'' = (V_0 - \epsilon)u_1, \quad (11)$$

$$(H_0 - \epsilon)u_2 = u_1 \quad \Rightarrow \quad (H_0 - \epsilon)^2 u_2 = 0 \quad \Rightarrow \quad u_2'' = (V_0 - \epsilon)u_2 - u_1, \quad (12)$$

i.e., we are employing a Jordan chain of length 2. Expressing this in matrix language [7, 8], this specific choice of basis $\{u_1, u_2\}$ means that, in the restriction to the two-dimensional subspace of functions belonging to $\text{Ker}(B_2^+)$, the initial Hamiltonian H_0 is represented by a matrix (H_0) having a non-trivial Jordan structure of standard type:

$$(H_0) = \begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix}. \quad (13)$$

Note that, if u_1 is given, then it is possible to determine the general solution u_2 to the second-order equation (12) and to calculate then explicitly $W(u_1, u_2)$ [21]. An alternative (and shorter) procedure runs as follows. First of all it is straightforward to show that

$$W'(u_1, u_2) = u_1 u_2'' - u_2 u_1'' = -u_1^2, \quad (14)$$

where we have used equations (11, 12). This implies that:

$$W(u_1, u_2) = w_0 - \int_{x_0}^x u_1^2(y) dy \equiv w(x), \quad (15)$$

where $x_0 \in (x_l, x_r)$.

Let us note that, in order that the new potential

$$V_2 = V_0 - 2\{\ln[W(u_1, u_2)]\}'' = V_0 - 2[\ln(w)]'' \quad (16)$$

has not additional singularities with respect to V_0 , then $w(x)$ must not have nodes in (x_l, x_r) . This can be achieved by choosing a Schrödinger seed solution u_1 such that [20]:

$$\lim_{x \rightarrow x_l} u_1 = 0, \quad \nu_- \equiv \int_{x_l}^{x_0} u_1^2(y) dy < \infty, \quad \text{or} \quad (17)$$

$$\lim_{x \rightarrow x_r} u_1 = 0, \quad \nu_+ \equiv \int_{x_0}^{x_r} u_1^2(y) dy < \infty. \quad (18)$$

With this choice, it turns out that $w(x)$ becomes nodeless either for $w_0 \leq -\nu_-$ in the first case or for $w_0 \geq \nu_+$ in the second one. Moreover, departing from the normalized bound states $\psi_n(x)$ of H_0 the normalized ones $\psi_n^{(2)}(x)$ of H_2 can be built up in the way:

$$\psi_n^{(2)}(x) = \frac{B_2^+ \psi_n(x)}{E_n - \epsilon}. \quad (19)$$

In addition, there is an eigenfunction of H_2 associated with ϵ which becomes as well square integrable (we are using here a notation for this state which is appropriate for the purpose of this paper):

$$u_1^{(2)}(x) \propto \frac{u_1(x)}{w(x)}. \quad (20)$$

Note that the confluent algorithm has been used to create bound states above the ground state energy of H_0 [20]. This possibility of spectral manipulation typically was outside the goals of the standard first-order SUSY QM. Moreover, the use of just one eigenfunction of H_0 in the confluent case is advantageous compared with the second-order SUSY QM with $\epsilon_1 \neq \epsilon_2$, which requires the knowledge of two Schrödinger seed solutions.

3 Hyperconfluent third-order SUSY QM: direct approach

In turn, let us analyze the hyperconfluent third-order SUSY QM, for which the three factorization energies converge to the same ϵ -value, namely $\epsilon_i \rightarrow \epsilon$, $i = 1, 2, 3$. Similarly as for the second-order case of Section 2, we are going to use here a Jordan chain of length 3 of generalized eigenfunctions $\{u_1, u_2, u_3\}$ such that u_1, u_2 obey equations (11,12) while u_3 satisfies

$$(H_0 - \epsilon)u_3 = u_2 \quad \Rightarrow \quad (H_0 - \epsilon)^3 u_3 = 0 \quad \Rightarrow \quad u_3'' = (V_0 - \epsilon)u_3 - u_2. \quad (21)$$

Equations (11,12,21) mean that in the three-dimensional subspace of functions belonging to $\text{Ker}(B_3^+)$ this choice of basis implies that the matrix representing to H_0 has a non-trivial Jordan structure of standard type ¹

$$(H_0) = \begin{pmatrix} \epsilon & 1 & 0 \\ 0 & \epsilon & 1 \\ 0 & 0 & \epsilon \end{pmatrix}. \quad (22)$$

Now, the hyperconfluent third-order SUSY partner Hamiltonians H_0 and H_3 are intertwined by the third-order operator B_3^+ in the way

$$H_3 B_3^+ = B_3^+ H_0, \quad (23)$$

where H_0 is given by equation (1), H_3 has the standard Schrödinger form

$$H_3 = -\frac{d^2}{dx^2} + V_3(x), \quad (24)$$

and V_3 is expressed in terms of the initial potential and the three seeds u_1, u_2, u_3 in the way:

$$V_3 = V_0 - 2\{\ln[W(u_1, u_2, u_3)]\}'', \quad (25)$$

with $W(u_1, u_2, u_3)$ denoting the Wronskian of u_1, u_2 and u_3 (we will give the explicit expression for B_3^+ in the next Section). A straightforward calculation leads to:

$$W(u_1, u_2, u_3) = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{vmatrix} = u_1'' W(u_2, u_3) - u_2'' W(u_1, u_3) + u_3'' W(u_1, u_2). \quad (26)$$

¹Note that the matrices (H_0) of equations (13) and (22) are non-hermitian despite H_0 is hermitian.

By using now equations (11,12,21) it turns out that:

$$W(u_1, u_2, u_3) = u_1 W(u_1, u_3) - u_2 W(u_1, u_2). \quad (27)$$

Recall that $W(u_1, u_2) = w(x)$ was calculated in a simple way in the previous Section; thus, a similar procedure to obtain $W(u_1, u_3)$ can be followed, leading to:

$$W(u_1, u_3) = w_1 - \int_{x_0}^x u_1(y) u_2(y) dy. \quad (28)$$

Given u_1 , and consequently the w of equation (15), it remains just to express u_2 in terms of them. Let us note first of all that

$$w = W(u_1, u_2) = u_1^2 \left(\frac{u_2}{u_1} \right)'. \quad (29)$$

Henceforth

$$u_2 = u_1 \left[\beta_1 + \int_{x_0}^x \frac{w(y)}{u_1^2(y)} dy \right]. \quad (30)$$

Thus, a straightforward calculation leads to

$$\int_{x_0}^x u_1(y) u_2(y) dy = w_0 \beta_1 - \beta_1 w(x) - w(x) \int_{x_0}^x \frac{w(y)}{u_1^2(y)} dy + \int_{x_0}^x \left[\frac{w(y)}{u_1(y)} \right]^2 dy. \quad (31)$$

By plugging equations (15,28,31) into equation (27) we arrive at:

$$W(u_1, u_2, u_3) = u_1 \left\{ f_0 - \int_{x_0}^x \left[\frac{w(y)}{u_1(y)} \right]^2 dy \right\} \equiv u_1 f, \quad (32)$$

with

$$f(x) = f_0 - \int_{x_0}^x \left[\frac{w(y)}{u_1(y)} \right]^2 dy, \quad (33)$$

and $f_0 = w_1 - w_0 \beta_1$. Finally, the potential of equation (25) becomes:

$$V_3(x) = V_0(x) - 2\{\ln[u_1(x)]\}'' - 2\{\ln[f(x)]\}'', \quad (34)$$

where $f(x)$ is given by equation (33).

4 Hyperconfluent third-order SUSY QM: iterative approach

We are going to apply now two consecutive SUSY transformations departing from the initial Hamiltonian H_0 : a confluent second-order one for generating V_2 from V_0 , which employs the two generalized eigenfunctions u_1, u_2 associated to ϵ satisfying equations (11,12) of Section 2; then a first-order transformation in order to obtain V_3 from V_2 , which makes use of the general solution of the stationary Schrödinger equation of H_2 associated to ϵ .

As for the confluent second-order transformation, we saw at Section 2 that the new potential V_2 is given by

$$V_2 = V_0 - 2\{\ln[W(u_1, u_2)]\}'' = V_0 - 2[\ln(w)]'', \quad (35)$$

where the Wronskian $W(u_1, u_2) = w(x)$ of the two generalized eigenfunctions u_1, u_2 of H_0 associated to ϵ is given by equation (15).

Concerning the first-order transformation, it turns out that H_2 and H_3 are intertwined by a first-order operator A_3^+ in the way:

$$H_3 A_3^+ = A_3^+ H_2, \quad (36)$$

where H_2 and H_3 are given by equations (4) and (24) respectively, and

$$A_3^+ = -\frac{d}{dx} + \ln[u^{(2)}]' = -\frac{d}{dx} + \frac{u^{(2)'}}{u^{(2)}}, \quad (37)$$

with $u^{(2)}$ being the general solution of the Schrödinger equation

$$H_2 u^{(2)} = \epsilon u^{(2)}.$$

From the results of Section 2 it is known that one solution is given by $u_1^{(2)} = u_1/w$ (see equation (20) and [20, 21]). The other linearly independent solution $u_2^{(2)}$ is found by asking that $W(u_1^{(2)}, u_2^{(2)}) = 1 = [u_1^{(2)}]^2 [u_2^{(2)}/u_1^{(2)}]'$, which immediately leads to $u_2^{(2)} = u_1^{(2)} \int_{x_0}^x dy / [u_1^{(2)}(y)]^2$. Thus, the solution $u^{(2)}$ we are looking for to implement the first-order transformation takes the form:

$$u^{(2)} = c_1 u_1^{(2)} + c_2 u_2^{(2)} = -c_2 \frac{u_1}{w} \left\{ -\frac{c_1}{c_2} - \int_{x_0}^x \left[\frac{w(y)}{u_1(y)} \right]^2 dy \right\}. \quad (38)$$

Hence, the final potential V_3 resulting from applying the first-order SUSY transformation to the Hamiltonian H_2 , when using the seed solution given in equation (38), becomes:

$$V_3 = V_2 - 2\{\ln[u^{(2)}]\}'' = V_0 - 2\{\ln[u_1]\}'' - 2 \left\{ \ln \left(-\frac{c_1}{c_2} - \int_{x_0}^x \left[\frac{w(y)}{u_1(y)} \right]^2 dy \right) \right\}'' . \quad (39)$$

Note that the two hyperconfluent third-order SUSY partner potentials $V_3(x)$ of $V_0(x)$ given by equations (34) and (39) are exactly the same if it is taken $f_0 = -c_1/c_2$.

We can give, finally, the explicit expression for the third-order operator B_3^+ intertwining the initial and final Hamiltonians H_0 and H_3 (see equation (23)):

$$B_3^+ = A_3^+ B_2^+, \quad (40)$$

where B_2^+ and A_3^+ are given by equations (6) and (37) respectively.

5 Non-singular transformations and bound states of H_3

As can be seen from equation (25), in order that the potential V_3 has no additional singularities compared with those of V_0 , the Wronskian $W(u_1, u_2, u_3)$ given in equation (32) should not have nodes inside \mathcal{D} . This implies that both functions u_1 and f in this factorized expression should be free of zeros in this domain, in particular the seed solution u_1 which automatically leads to the restriction $\epsilon \leq E_0$, where E_0 is the ground state energy of H_0 . Moreover, for the second factor $f(x)$ of equation (33) to be nodeless, the function w/u_1 should vanish either to the left edge x_l of \mathcal{D} or to the right one x_r . Here we are going to discuss in detail just the first case; the second one can be addressed in a similar way.

Let us choose first of all a nonphysical Schrödinger seed solution u_1 without nodes in \mathcal{D} , obeying equation (11) for $\epsilon < E_0$. Moreover, it is supposed that u_1 satisfies as well equation (17). Since w/u_1 should vanish for $x \rightarrow x_l$, we must have

$$\lim_{x \rightarrow x_l} w = w_0 + \nu_- = 0, \quad (41)$$

which implies that w_0 has to be taken as

$$w_0 = -\nu_- = - \int_{x_l}^{x_0} u_1^2(y) dy. \quad (42)$$

Therefore:

$$w(x) = - \int_{x_l}^x u_1^2(y) dy. \quad (43)$$

With this specific choice of u_1 and w , for most of the typical quantum mechanical problems it turns out that:

$$\lim_{x \rightarrow x_l} \frac{w(x)}{u_1(x)} = 0 \quad \text{and} \quad \lim_{x \rightarrow x_r} \left| \frac{w(x)}{u_1(x)} \right| = \infty. \quad (44)$$

Hence, the domain of the parameter f_0 such that $f(x)$ is nodeless in (x_l, x_r) is given by

$$f_0 < -\sigma_- = - \int_{x_l}^{x_0} \frac{w^2(y)}{u_1^2(y)} dy. \quad (45)$$

Let us note that in this f_0 -domain the third-order intertwining operator B_3^+ of equation (40) transforms the normalized eigenfunctions ψ_n of H_0 into normalized eigenfunctions $\psi_n^{(3)}$ of H_3 in the way:

$$\psi_n^{(3)}(x) = \frac{B_3^+ \psi_n}{\sqrt{(E_n - \epsilon)^3}}. \quad (46)$$

Moreover, the eigenfunction $\psi_\epsilon^{(3)}$ of H_3 associated to the eigenvalue ϵ (compare equation (38)),

$$\psi_\epsilon^{(3)}(x) \propto \frac{1}{u^{(2)}(x)} \propto \frac{w(x)}{u_1(x)f(x)}, \quad (47)$$

turns out to be square-integrable in \mathcal{D} , which implies that the spectrum of H_3 becomes

$$\text{Sp}(H_3) = \{\epsilon\} \cup \text{Sp}(H_0). \quad (48)$$

Note that, when $f_0 \rightarrow -\sigma_-$, the hyperconfluent third-order transformation remains non-singular but the eigenstate $\psi_\epsilon^{(3)}$ is not longer square-integrable. Thus, in this limit the two Hamiltonians H_3 and H_0 become isospectral.

On the other hand, for $u_1(x)$ being chosen as the normalized ground state eigenfunction $\psi_0(x)$ of H_0 associated to E_0 , it turns out that the previous equations (41-47) remain valid, the only difference is that now

$$\nu_- = \int_{x_l}^{x_0} u_1^2(y) dy < 1.$$

Thus, for f_0 satisfying equation (45), it turns out that $\epsilon = E_0 \in \text{Sp}(H_3)$, which implies that

$$\text{Sp}(H_3) = \text{Sp}(H_0), \quad (49)$$

i.e., the transformation is again strictly isospectral. However, when $f_0 \rightarrow -\sigma_-$ the $\psi_\epsilon^{(3)}$ of equation (47) is not longer normalizable, meaning that in this limit $\epsilon = E_0 \notin \text{Sp}(H_3)$, namely,

$$\text{Sp}(H_3) = \text{Sp}(H_0) - \{E_0\}. \quad (50)$$

In this case, through the hyperconfluent third-order SUSY transformation somehow we ‘delete’ the ground state energy of H_0 for generating H_3 .

6 Examples

Let us apply next the previous formalism to two physically interesting examples, the free particle and the Coulomb potential.

6.1 Free particle

The general solution of the stationary Schrödinger equation (11) for the free particle with a negative factorization energy $\epsilon = -k^2$, $k > 0$ (for which $V_0(x) = 0$) is given by:

$$u_1(x) = Ae^{kx} + Be^{-kx}. \quad (51)$$

In order to apply our method, let us use a nonphysical seed solution $u_1(x)$ satisfying equation (17) for $x_l = -\infty$, i.e., let us make in equation (51) $B = 0$ and $A = 1$ so that:

$$u_1(x) = e^{kx}. \quad (52)$$

With this choice, the calculation of equation (43) leads to:

$$w(x) = -\frac{e^{2kx}}{2k}. \quad (53)$$

Moreover, the evaluation of equation (33) with $x_0 = 0$ produces:

$$f(x) = f_0 + \frac{1 - e^{2kx}}{8k^3}. \quad (54)$$

Note that this function does not have nodes for

$$f_0 < -\sigma_- = -\frac{1}{8k^3}.$$

Hence, it is convenient to reparametrize this domain in the way:

$$f_0 = -\frac{1}{8k^3} - \frac{e^{2kx_1}}{8k^3}, \quad (55)$$

where $x_1 \in (-\infty, \infty)$. Thus, it is straightforward to show that:

$$f(x) = -\frac{e^{k(x+x_1)}}{4k^3} \cosh[k(x - x_1)]. \quad (56)$$

Finally, by plugging equations (52,56) into equation (34), the hyperconfluent third-order SUSY partner potential of the free particle turns out to be:

$$V_3(x) = -2k^2 \text{sech}^2[k(x - x_1)]. \quad (57)$$

This is the well know Pöschl-Teller potential with one bound state at $\epsilon = E_0 = -k^2$, which has been also derived through first-order SUSY (see e.g. [2], page 30) and confluent second-order SUSY techniques [20].

6.2 Coulomb potential

Working in spherical coordinates, separating the angular ones θ, ϕ , and making $\hbar = e = m = 1$, the three-dimensional stationary Schrödinger equation for the Coulomb potential $-e^2/r$ leads to a one-dimensional problem characterized by the effective potential

$$V_0(r) = -\frac{2}{r} + \frac{\ell(\ell+1)}{r^2}, \quad (58)$$

where $0 \leq r < \infty$, $\ell = 0, 1, \dots$. The discrete energy levels E_n of H_0 , for a fixed value of ℓ , take the form $E_n = -1/(n + \ell + 1)^2$, $n = 0, 1, 2, \dots$. In order to apply our method, let us employ here the normalized ground state eigenfunction,

$$u_1(r) = \frac{1}{(\ell+1)\sqrt{(2\ell+1)!}} \left(\frac{2r}{\ell+1}\right)^{\ell+1} e^{-\frac{r}{\ell+1}}, \quad (59)$$

associated to the eigenvalue $E_0 = -1/(\ell+1)^2$. Let us start by calculating the $w(r)$ of equation (43) with $r_l = 0$, which leads to:

$$w(r) = -\frac{\gamma(2\ell+3, \frac{2r}{\ell+1})}{(2\ell+2)!} = e^{-\frac{2r}{\ell+1}} \sum_{k=0}^{2\ell+2} \frac{1}{k!} \left(\frac{2r}{\ell+1}\right)^k - 1 = -e^{-\frac{2r}{\ell+1}} \sum_{k=2\ell+3}^{\infty} \frac{1}{k!} \left(\frac{2r}{\ell+1}\right)^k, \quad (60)$$

$\gamma(a, x)$ being an incomplete Gamma function. Using this result and the expression for $u_1(r)$ of equation (59) it turns out that:

$$\frac{w(r)}{u_1(r)} = -(\ell+1)\sqrt{(2\ell+1)!} e^{-\frac{r}{\ell+1}} \sum_{k=2\ell+3}^{\infty} \frac{1}{k!} \left(\frac{2r}{\ell+1}\right)^{k-\ell-1}, \quad (61)$$

which vanishes for $r \rightarrow 0$, as required. The calculation of the $f(r)$ of equation (33) with $r_0 = 0$ produces now:

$$\begin{aligned} f(r) &= f_0 - \frac{(\ell+1)^3(2\ell+1)!}{2} \sum_{k=2\ell+3}^{\infty} \sum_{m=2\ell+3}^{\infty} \frac{\gamma(k+m-2\ell-1, \frac{2r}{\ell+1})}{k! m!} \\ &= f_0 - \frac{\gamma(2\ell+3, \frac{2r}{\ell+1})}{2\Gamma(2\ell+4)} r^2 {}_2F_2\left(1, 2; 3, 2\ell+4; \frac{2r}{\ell+1}\right) \\ &\quad + \frac{(\ell+1)^2}{4} \sum_{m=0}^{\infty} \frac{\gamma(m+2\ell+5, \frac{2r}{\ell+1})}{(m+2)(m+2\ell+3)!}, \end{aligned} \quad (62)$$

which is nodeless in $(0, \infty)$ for $f_0 \leq 0$. The hyperconfluent third-order SUSY partner of the effective potential (58) becomes finally:

$$V_3(r) = -\frac{2}{r} + \frac{(\ell+1)(\ell+2)}{r^2} + 2 \left[\frac{w^2(r)}{f(r)u_1^2(r)} \right]', \quad (63)$$

where $u_1(r)$, $w(r)$ and $f(r)$ are given by equations (59,60) and (62) respectively.

The first two terms of equation (63) correspond to an effective potential different from the initial one (compare equation (58)). This difference is also reflected in the energy levels of a potential composed only of these two terms, which are given by $E_n = -1/(n + \ell + 1)^2$, $n = 1, 2, \dots$. Thus,

it is natural to interpret that the third term of equation (63) is the main responsible of supporting the ground state energy of $V_3(r)$ at $E_0 = -1/(\ell + 1)^2$.

Let us note that the family of hyperconfluent third-order SUSY partner potentials given by equation (63) is different from the ones which have been derived either by first-order SUSY [15, 16, 30–32] or by second-order SUSY transformations [15, 16, 21] (just compare the centrifugal terms of each family).

In particular, for $\ell = 0$ it turns out that, departing from the Coulomb potential without centrifugal term, $V_0(r) = -2/r$, we arrive at a new one-dimensional potential with a non-trivial centrifugal term given by

$$V_3(r) = -\frac{2}{r} + \frac{2}{r^2} + 2 \left[\frac{w^2(r)}{f(r)u_1^2(r)} \right]', \quad (64)$$

where now

$$u_1(r) = 2re^{-r}, \quad w(r) = -\frac{1}{2}\gamma(3, 2r) = (2r^2 + 2r + 1)e^{-2r} - 1, \quad (65)$$

$$f(r) = f_0 - \frac{1}{12}\gamma(3, 2r)r^2 {}_2F_2(1, 2; 3, 4; 2r) + \frac{1}{4} \sum_{m=0}^{\infty} \frac{\gamma(m+5, 2r)}{(m+2)(m+3)!}. \quad (66)$$

As an illustration, the isospectral potentials $V_0(r) = -2/r$ and the $V_3(r)$ of equations (64-66) for $f_0 = -1/10$ as functions of r are shown in figure 1. The corresponding energy levels of H_3 and H_0 are given by $E_n = -1/(n+1)^2$, $n = 0, 1, 2, \dots$

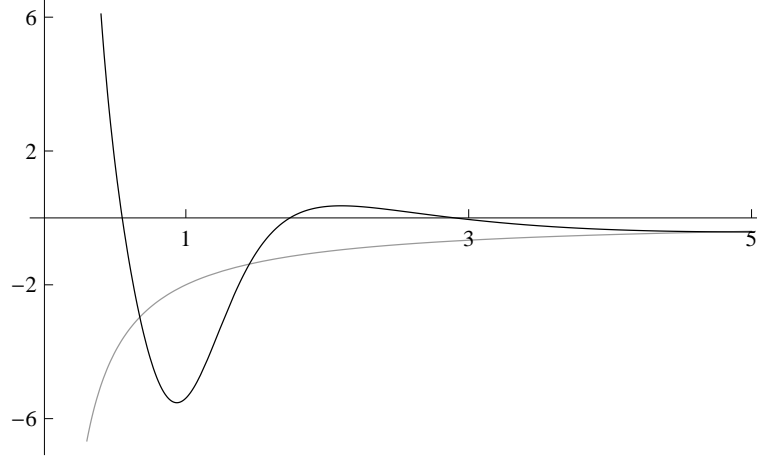


Figure 1: Coulomb potential $V_0(r) = -2/r$ (gray curve) and its hyperconfluent third-order SUSY partner $V_3(r)$ given by equations (64-66) for $f_0 = -1/10$.

Let us note that the well arising in $V_3(r)$ at the neighborhood of $r \approx 1$ is induced by the third term of equation (64), which is also the responsible of supporting the ground state energy at $E_0 = -1$. Let us recall that this level was not present in the effective potential of equation (58) with $\ell = 1$.

7 Conclusions

In this paper we have addressed the hyperconfluent third-order SUSY QM through two different (but equivalent) approaches, namely, direct and iterative one. It was found the explicit expression

for the Wronskian, the most relevant quantity which determines the form of the new potentials, the eigenfunctions of the associated Hamiltonians, etc.

The requirements for the seed solution to produce non-singular SUSY transformations were as well explicitly determined. Note that, from considerations taking into account the order of the transformation, through the hyperconfluent third-order SUSY QM one obtains a three-parametric family of potentials (for a fixed factorization energy). However, since we had to impose two requirements on the solutions employed in the iterative approach, it turns out that the non-singular potentials for $r \in (0, \infty)$ belong just to a one-parametric subset of the general three-parametric family which one is able to build up.

Our general procedure was illustrated by means of the free particle and the Coulomb potential. In particular, the last case illustrates clearly that the non-singular one-parametric family of potentials derived through the hyperconfluent third-order SUSY QM is different either from the set which can be achieved from a first-order SUSY transformation [15, 16, 30–32] or from the one which can be generated through the confluent second-order SUSY transformation [21].

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